# ANNEXE TO THE GALLERY: AN ADDENDUM TO "ADDITIVE REPRESENTATION IN THIN SEQUENCES, VIII: DIOPHANTINE INEQUALITIES IN REVIEW"

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We exploit recent developments in Vinogradov's mean value theorem in order to improve some of our earlier conclusions concerning Diophantine inequalities.

## 11. Downloading updates

The purpose of our eighth exhibition devoted to additive representation in thin sequences* was to review the implications for the theory of Diophantine inequalities of our methods devoted to exceptional sets in thin sequences. A problem in the final stage of production of [2] led to confusion in the numbering and cross-referencing of equation numbers in the text, a problem which occurred too late for the authors to address. The editors of

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*in order to make for concise referencing to part VIII of the ARTS sequence, we adopt the convention that sections, theorems, equations, and so on, referenced in this addendum refer to the restored version of [2] contained in this volume. Though separate in order to preserve the intellectual integrity of the original publication, we regard this addendum as §11 of the present restoration. The single exception to this rule is in the bibliography, where we have provided a complete list for ease of reference.

the present volume have agreed to reproduce the correct version of [2] in its entirety, a very generous gesture for which the authors are extremely grateful. With the unveiling in this volume of the restoration of [2], we have the opportunity to reflect on such improvements and enhancements of our earlier work as are made available by technological advances over the intervening two years. In order to preserve the intellectual integrity of the original paper, we download such updates in the present addendum rather than muddy the historical waters by revising the original paper.

Even in the short space of time that has elapsed since the original publication of [2], advances in our understanding of Weyl sums introduced in [5,6] permit fairly substantial improvements to be made in the estimates stemming from Lemma 5.3. We first recall the notation introduced in the context of the latter lemma. Let $\phi \in \mathbb{R}[t]$ denote a polynomial of degree $d \geq 3$. Let $s \geq 2$, and write $U_{\phi,s}(M)$ for the number of integral solutions of the inequality

$$\left| \sum_{j=1}^{s} (\phi(x_j) - \phi(y_j)) \right| < 1,$$

with $1 \leq x_j, y_j \leq M$.

**Lemma 11.1.** *When $s \geq d^2$, one has $U_{\phi,s}(M) \ll M^{2s-d}$.*

**Proof.** Write $J_{d,s}(M)$ for the number of solutions of the Diophantine system

$$\sum_{i=1}^{s} (x_i^j - y_i^j) = 0 \quad (1 \leq j \leq d),$$

with $1 \leq x_i, y_i \leq M$. Then it follows from Lemma 5.3 that $U_{\phi,s}(M) \ll M^{\frac{1}{2}d(d-1)} J_{d,s}(M)$. The proof of [5, Theorem 1.2] presented in §9 of the latter source, when combined with [6, Theorem 1.1], shows that $J_{d,s}(M) \ll M^{2s-\frac{1}{2}d(d+1)}$ whenever $s \geq d^2$. Thus we deduce that under the latter condition on $s$, one has

$$U_{\phi,s}(M) \ll M^{\frac{1}{2}d(d-1)} \cdot M^{2s-\frac{1}{2}d(d+1)} \ll M^{2s-d}.$$

This completes the proof of the lemma.                                            □

We apply this new lemma in two directions within this addendum. First we analyse the value distribution of binary linear forms in prime numbers, deriving an improvement of Theorem 1.6. In order to describe this conclusion in a self-contained manner, we recall some notation from our earlier

work. When $\lambda_1, \lambda_2$ are positive numbers, and $0 < \tau \leq 1$ and $\nu > 0$, we denote by $\sigma_\lambda(\tau, \nu)$ the number of prime solutions $p_1, p_2$ of the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 - \nu| < \tau,$$

with each solution $p_1, p_2$ counted with weight $(\log p_1)(\log p_2)$.

**Theorem 11.2.** *Let $\lambda_1, \lambda_2$ denote positive real numbers such that $\lambda_1/\lambda_2$ is an algebraic irrational. Fix $0 < \tau < 1$ and $A \geq 1$. Let $\phi$ denote a positive polynomial of degree $d$, and let $E_\phi(N)$ denote the number of integers $n$ with $1 \leq n \leq N$ for which the inequality*

$$\left|\sigma_\lambda(\tau, \nu) - \frac{2\tau\nu}{\lambda_1\lambda_2}\right| > \frac{\tau\nu}{(\log N)^A}$$

*holds with $\nu = \phi(n)$. Then*

$$E_\phi(N) \ll N^{1-1/(6d)+\varepsilon}.$$

**Proof.** The conclusion of Lemma 11.1 confirms the upper bound (5.25) with $s = d^2$. The estimate for $E_\phi(N)$ claimed in the conclusion of the theorem therefore follows from Lemma 5.4, just as in the proof of Theorem 1.6 presented following the statement of the aforementioned lemma. $\square$

In the situation where $\phi$ is an integral polynomial, a conclusion of similar shape to that of Theorem 11.2 is available from our earlier work [1], as a consequence of Ford's method [3] of bounding mean values of one-dimensional Weyl sums via Vinogradov's mean value theorem. This being unavailable for exponential sums over polynomials $\phi$ not equivalent to integral polynomials, we were forced in Theorem 1.6 to content ourselves with the weaker conclusion $E_\phi(N) \ll N^{1-\delta/(d \log d)}$ for a suitable $\delta > 0$. The stronger conclusion provided by Theorem 11.2 is made possible by the sharp versions of Vinogradov's mean value theorem very recently made available in [5,6].

We turn next to our second and final consequence of Lemma 11.1, an improvement of Theorem 1.10. We recall the notation surrounding the statement of the latter theorem. We consider a set of non-zero positive polynomials

$$\phi_j(t) = \sum_{l=1}^{d_j} \lambda_{jl} t^l \quad (1 \leq j \leq s)$$

without constant terms, and of degree $d_j \geq 1$. In particular, we suppose that $\lambda_j = \lambda_{jd_j}$ is non-zero. Denote by $\rho_\phi(\tau, \nu)$ the number of solutions of

$$|\phi_1(x_1) + \ldots + \phi_s(x_s) - \nu| < \tau,$$

in positive integers.

**Theorem 11.3.** *Let $s$ be a natural number, and suppose that $d_1, \ldots, d_s$ are natural numbers satisfying the condition that*

$$\sum_{i=1}^{s} d_i^{-2} > 2.$$

*Suppose in addition that $\phi_1, \ldots, \phi_s$ are polynomials of respective degrees $d_1, \ldots, d_s$, satisfying the conditions imposed in the preamble to the statement of this theorem, and subject also to the condition that at least one of the ratios $\lambda_{jl}/\lambda_{km}$ is irrational. Then*

$$\rho_\phi(\tau, \nu) = c(\phi)\tau\nu^{D-1} + o(\nu^{D-1}),$$

*in which $D = d_1^{-1} + \ldots + d_s^{-1}$ and*

$$c(\phi) = \frac{2\Gamma(1 + d_1^{-1}) \ldots \Gamma(1 + d_s^{-1})}{\Gamma(D)\lambda_1^{1/d_1} \ldots \lambda_s^{1/d_s}}.$$

**Proof.** We apply the argument of §7. We define $X_j$ to be the unique positive solution of $\phi_j(X_j) = \nu$, and then define the Weyl sums

$$f_j(\alpha) = \sum_{x \leq 2X_j} e(\alpha\phi_j(x)).$$

In addition, put $t_j = d_j^2$ ($1 \leq j \leq s$). Then on considering the number of integral solutions of the underlying Diophantine inequality, it follows from Lemma 11.1 that for $1 \leq j \leq s$ one has

$$\int |f_j(\alpha)|^{2t_j} \, \mathrm{d}_\tau\alpha \ll X_j^{2t_j - d_j} \ll X_j^{2t_j}\nu^{-1}.$$

Write $\kappa = d_1^{-2} + \ldots + d_s^{-2}$, and then put $\theta = 1 - 2/\kappa$. Then we deduce via Hölder's inequality that

$$\int |f_1(\alpha) \ldots f_s(\alpha)|^{1-\theta} \, \mathrm{d}_\tau\alpha \ll \prod_{j=1}^{s}\left(\int |f_j(\alpha)|^{2t_j} \, \mathrm{d}_\tau\alpha\right)^{(1-\theta)/(2t_j)}$$

$$\ll \prod_{j=1}^{s}\left(X_j^{2t_j}\nu^{-1}\right)^{(1-\theta)/(2t_j)}.$$

Observe that

$$\sum_{j=1}^{s}(1-\theta)/(2t_j) = \tfrac{1}{2}(1-\theta)\kappa = 1.$$

Thus, we deduce that for any measurable set $\mathfrak{B} \subseteq \mathbb{R}$, one has

$$\int_{\mathfrak{B}} f_1(\alpha) \ldots f_s(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha$$

$$\ll \left( \sup_{\alpha \in \mathfrak{B}} |f_1(\alpha) \ldots f_s(\alpha)|^\theta \right) \left( (X_1 \ldots X_s)^{1-\theta} \nu^{-1} \right). \quad (11.1)$$

Write

$$d = \max_{1 \leq j \leq s} d_j, \quad Y = \nu^{1/(2d)-1}, \quad \mathfrak{C} = [-Y, Y] \quad \text{and} \quad \mathfrak{c} = \mathbb{R} \setminus \mathfrak{C}.$$

Then it follows from Lemma 7.1 that there exists a monotone function $T(\nu)$, with $T(\nu) \to \infty$ as $\nu \to \infty$, such that

$$\sup_{Y \leq |\alpha| \leq T(\nu)} |f_1(\alpha) f_2(\alpha) \ldots f_s(\alpha)| = o(X_1 X_2 \ldots X_s).$$

It therefore follows from (11.1) that

$$\int_{Y \leq |\alpha| \leq T(\nu)} f_1(\alpha) f_2(\alpha) \ldots f_s(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha = o(X_1 X_2 \ldots X_s \nu^{-1}). \quad (11.2)$$

On the other hand, the estimate

$$\int_{|\alpha| > T(\nu)} f_1(\alpha) f_2(\alpha) \ldots f_s(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha \ll X_1 X_2 \ldots X_s \nu^{-1} T(\nu)^{-1} \quad (11.3)$$

is a consequence of the trivial bound $|f_j(\alpha)| \leq X_j$ $(1 \leq j \leq s)$ in combination with Lemma 11 of Freeman [4]. By combining (11.2) and (11.3), we conclude that

$$\int_{\mathfrak{c}} f_1(\alpha) \ldots f_s(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha = o(\nu^{D-1}). \quad (11.4)$$

The discussion of §7.1 leading to equation (7.3), together with the discussion of §7.2 therein, shows that whenever $s \geq 2d$, then

$$\rho_\phi(\tau, \nu) - c(\phi) \tau \nu^{D-1} = \int_{\mathfrak{c}} f_1(\alpha) \ldots f_s(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha + o(\nu^{D-1}),$$

whence by (11.4) one sees that

$$\rho_\phi(\tau, \nu) = c(\phi) \tau \nu^{D-1} + o(\nu^{D-1}).$$

This completes the proof of the theorem. $\qquad\qquad\square$

A similar conclusion to that of Theorem 11.3 was obtained in Theorem 1.10, though subject to the condition that $s \geq s_1(d) \sim 2d^2 \log d$, where

$d = \max_{1 \leq j \leq s} d_j$. The conclusion of Theorem 11.3 would imply the same result whenever $s \geq 2d^2$, the improvement here stemming from recent progress on Vinogradov's mean value theorem (see [5,6]). Aside from improving considerably the constraint on $s$ in terms of the maximal degree $d$, Theorem 11.3 also offers the flexibility to take advantage of smaller values of the exponents $d_j$ by comparison with $d$.

## References

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